Decision Modelling II

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MCDM Summer School 2018

Outline

1. Preferences Handling
   - Problem Setting
   - Basics
   - Preference Modeling
   - Preference Aggregation
2. Social Choice Theory
3. Borda and Condorcet
   - The Borda path
   - The Condorcet path
4. Conclusions

Preferences

- Preferences are “rational” desires.
- Preferences are at the basis of any decision aiding activity.
- There are no decisions without preferences.
- Preferences, Values, Objectives, Desires, Utilities, Beliefs, ...

Preference Statements:

- I like red shoes.
- I do not like brown sugar.
- I prefer Maria to Mario.
- I do not want tea with milk.
- Cost is more important than safety.
- I prefer flying to Athens than having a suite at Istanbul.
- I prefer red wine only if there is no fish plate available.

Preference Statements:

Four issues:

Relative vs Absolute statements
Single vs Multi-attribute statements
Positive vs Negative statements
First vs Second order statements

What are the problems?

- How to learn preferences?
- How to model preferences?
- How to aggregate preferences?
- How to use preferences for recommending?

Binary relations

- $\succeq$: binary relation on a set (A).
- $\succeq \subseteq A \times A$ or $A \times P \cup P \times A$.
- $\succeq$ is reflexive.

What is that?

If $x \succeq y$ stands for x is at least as good as y, then the asymmetric part of $\succeq$ ($\succ$: $x \succeq y \;\wedge\; \neg(y \succeq x)$) stands for strict preference. The symmetric part stands for indifference ($\sim_1$: $x \succeq y \;\wedge\; y \succeq x$) or incomparability ($\sim_2$: $\neg(x \succeq y) \;\wedge\; \neg(y \succeq x)$).

More binary relations

- We can further separate the asymmetric (symmetric) part in more relations representing hesitation or intensity of preference.

$$\succ = \succ_1 \cup \succ_2 \cdots \succ_n$$

- We can get rid of the symmetric part since any symmetric relation can be viewed as the union of two asymmetric relations and the identity.
- We can also have valued relations such that: $v(x \succ y) \in [0, 1]$

Binary relations properties

Binary relations have specific properties such as:

- Irreflexive: $\forall x \;\; \neg(x \succ x)$;
- Asymmetric: $\forall x, y \;\; x \succ y \; \rightarrow \; \neg(y \succ x)$;
- Transitive: $\forall x, y, z \;\; x \succ y \; \wedge \; y \succ z \; \rightarrow \; x \succ z$;
- Ferrers; $\forall x, y, z, w \;\; x \succ y \; \wedge \; z \succ w \; \rightarrow \; x \succ w \; \vee \; z \succ y$;

Numbers

$$x \succeq y \quad \Leftrightarrow \quad \Phi(x, y) \geq 0$$

where:

$\Phi : A \times A \mapsto \mathbb{R}$. Simple case $\Phi(x, y) = f(x) - f(y); \quad f : A \mapsto \mathbb{R}$

Preference Structures

A preference structure

is a collection of binary relations $\sim_1, \cdots \sim_m, \succ_1, \cdots \succ_n$ such that:

- they are pair-disjoint;
- $\sim_1 \cup \cdots \sim_m \cup \succ_1 \cup \cdots \succ_n = A \times A$;
- $\sim_i$ are symmetric and $\succ_j$ are asymmetric;
- possibly they are identified by their properties.

$\sim_1, \sim_2, \succ$ Preference Structures

Independently from the nature of the set A (enumerated, combinatorial etc.), consider $x, y \in A$ as whole elements. Then:

If $\succeq$ is a weak order then:

$\succ$ is a strict partial order, $\sim_1$ is an equivalence relation and $\sim_2$ is empty.

If $\succeq$ is an interval order then:

$\succ$ is a partial order of dimension two, $\sim_1$ is not transitive and $\sim_2$ is empty.

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$\sim_1, \sim_2, \succ_1 \succ_2$ Preference Structures

If $\succeq$ is a *PQI* interval order then:

$\succ_1$ is transitive, $\succ_2$ is quasi transitive, $\sim_1$ is asymmetrically transitive and $\sim_2$ is empty.

If $\succeq$ is a pseudo order then:

$\succ_1$ is transitive, $\succ_2$ is quasi transitive, $\sim_1$ is non transitive and $\sim_2$ is empty.

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What characterises such structures?

Characteristic Properties

Weak Orders are complete and transitive relations.
Interval Orders are complete and Ferrers relations.

Numerical Representations

w.o. $\Leftrightarrow$ $\exists f : A \mapsto \mathbb{R} : x \succeq y \leftrightarrow f(x) \geq f(y)$

i.o. $\Leftrightarrow$ $\exists f, g : A \mapsto \mathbb{R} : f(x) > g(x); x \succeq y \leftrightarrow f(x) \geq g(y)$

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More about structures

Characteristic Properties

PQI Interval Orders are complete and generalised Ferrers relations.
Pseudo Orders are coherent bi-orders.

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What if A is multi-attribute described?

$$x = \langle x_1 \cdots x_n \rangle \quad y = \langle y_1 \cdots y_n \rangle$$

$$x \succeq y \quad \Leftrightarrow \quad \Phi([u_1(x_1) \cdots u_n(_n)], [u_1(y_1) \cdots u_n(y_n)] \geq 0$$

A special case is when Φ is increasing to its first n arguments and decreasing to the following n arguments: it then can be an additive function. See more in conjoint measurement theory.

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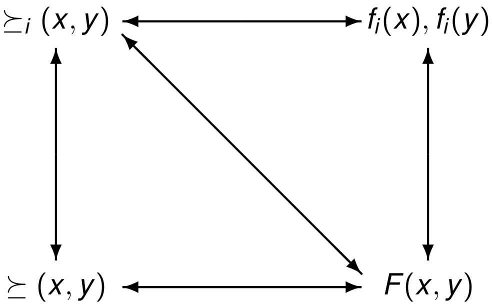
The Problem

Suppose we have n ordering relations $\succeq_1 \cdots \succeq_n$ on the set A. We are looking for an overall ordering relation $\succeq$ on A "representing" the different orders.



Borda vs. Condorcet

Four candidates and seven examiners with the following preferences.

	a	b	c	d	e	f	g
A	1	2	4	1	2	4	1
B	2	3	1	2	3	1	2
C	3	1	3	3	1	2	3
D	4	4	2	4	4	3	4

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B(x)
15
14
16
25

The Borda count gives B$>$A$>$C$>$D

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15

If D is not there then A$>$B$>$C, instead of B$>$A$>$C

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The Condorcet principle gives A>B>C>A !!!!

Arrow's Theorem

Given N rational voters over a set of more than 3 candidates can we found a social choice procedure resulting in a social complete order of the candidates such that it respects the following axioms?

- Universality: the method should be able to deal with any configuration of ordered lists;
- Unanimity: the method should respect a unanimous preference of the voters;
- Independence: the comparison of two candidates should be based only on their respective standings in the ordered lists of the voters.

YES!

There is only one solution: the dictator!!

If we add no-dictatorship among the axioms then there is no solution.

Gibbard-Satterthwaite's Theorem

When the number of candidates is larger than two, there exists no aggregation method satisfying simultaneously the properties of universal domain, non-manipulability and non-dictatorship.

Why MCDA is not Social Choice?

Social Choice	MCDA
Total Orders	Any type of order
Equal importance of voters	Variable importance of criteria
As many voters as necessary	Few coherent criteria
No prior information	Existing prior information

Counting values

$$x \succeq y \Leftrightarrow \sum_j r_j(x) \geq \sum_j r_j(y)$$

What do we need to know?

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the primitives: $\succeq_j \subseteq A \times A$
Differences of preferences:
- $(xy)_1 \succcurlyeq (zw)_1$
- $(xy)_1 \succcurlyeq (zw)_2$

How do we learn that?

- Directly through a standard protocol.
- Indirectly:
  - through pairwise comparisons (AHP, MACBETH etc.);
  - through learning from examples (regression, rough sets, decision trees etc.).

Is this sufficient?

NO!

Are preferences independent?
$r \succ w$
$f \succ m$
But *rf* is not better than *wf* ...

Non linear aggregation procedures

What is the output?

- Value functions on each criterion.
- A global value function.
- Rankings, choices, but also ratings if relevant reference points are provided on the value function.

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the primitives: $\succeq_j \subseteq A \times A$
An ordering relation on $2^{\succeq_j}$

How do we learn that?

- Preferences are “given”.
- Preferences on $2^{\succeq_j}$:
  - directly;
  - coalition games;
  - learning from examples.

Is this sufficient?

NO!

- The relation $\succeq$ is not an ordering relation.
- We need to construct an ordering relation $\succcurlyeq$ "as near as possible" to $\succeq$.
- In order to do so we transform the graph induced by $\succeq$.

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General idea: coalitions

Given a set A and a set of $\succeq_i$ binary relations on A (the criteria) we define:

$$x \succeq y \Leftrightarrow C^+(x,y) \trianglerighteq C^+(y,x) \text{ and } C^-(x,y) \trianglelefteq C^-(y,x)$$

where:
- $C^+(x,y)$: "importance" of the coalition of criteria supporting x wrt to y.
- $C^-(x,y)$: "importance" of the coalition of criteria against x wrt to y.

How it works? 1

Additive Positive Importance

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$$C^+(x,y) = \sum_{j \in J^\pm} w_j^+$$

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Then we can fix a majority threshold δ and have

$$x \succeq^+ y \Leftrightarrow C^+(x, y) \geq \delta$$

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Where "positive importance" comes from?

How it works? 2

Max Negative Importance

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$$C^-(x,y) = \max_{j \in J^-} w_j^-$$

where:
w_j^-: "negative importance" of criterion i
$J^- = \{h_j : v_j(x,y)\}$

How it works? 2

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where:
w_j^-: “negative importance” of criterion i
$J^- = \{h_j : v_j(x, y)\}$

Then we can fix a veto threshold γ and have

$$x \succeq^- y \Leftrightarrow C^-(x, y) \geq \gamma$$

How it works? 2

Max Negative Importance

$$C^{-}(x,y) = \max_{j \in J^{-}} w_j^{-}$$

where:
w_j^{-}: "negative importance" of criterion i
$J^{-} = \{h_j : v_j(x,y)\}$

Then we can fix a veto threshold γ and have

$$x \succeq^{-} y \Leftrightarrow C^{-}(x,y) \geq \gamma$$

Where "negative importance" comes from?

Example

The United Nations Security Council

Positive Importance

15 members each having the same positive importance $w_j^+ = \frac{1}{15}$, $\delta = \frac{9}{15}$.

Negative Importance

10 members with 0 negative importance and 5 (the permanent members) with $w_j^- = 1$, $\gamma = 1$.

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Outranking Principle

$$x \succeq y \Leftrightarrow x \succeq^{+} y \text{ and } \neg(x \succeq^{-} y)$$

Thus:

$$x \succeq y \Leftrightarrow C^{+}(x,y) \geq \delta \wedge C^{-}(x,y) < \gamma$$

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NB

The relation $\succeq$ is not an ordering relation. Specific algorithms are used in order to move from $\succeq$ to an ordering relation $\succcurlyeq$

What is importance?

Where w_j^+, w_j^- and δ come from?

Further preferential information is necessary, usually under form of multi-attribute comparisons. That will provide information about the decisive coalitions.

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Example

Given a set of criteria and a set of decisive coalitions ($J^{\pm}$) we can solve:

$$\max \delta$$
subject to
$$\sum_{j \in J^{\pm}} w_j \geq \delta$$
$$\sum_j w_j = 1$$

And the final ranking?

- $x \succcurlyeq y \;\Leftrightarrow\; o(x) - i(x) \geq o(y) - i(y)$
- Recursively constructing $\succcurlyeq$:
  - $[x]_1 = \{x \in A : \;\neg\; \exists y \;\; y \succeq x\}$
    $[x]_i = \{x \in A \setminus \cup_{i-1}[x] : \;\neg\; \exists y \;\; y \succeq x\}$
  - $[x]_n = \{x \in A : \;\neg\; \exists y \;\; x \succeq y\}$
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Rating

What if we have preference relations $\succeq_j \subseteq A \times P \cup P \times A$?
The global preference relation remains the same.

- pessimistic rating
  - x is iteratively compared with $p_t \cdots p_1$,
  - as soon as $(x \succeq p_h)$ is established, assign x to category c_h.
- optimistic rating
  - x is iteratively compared with $p_1 \cdots p_t$,
  - as soon as is established $(p_h \succeq x) \wedge \neg (x \succeq p_h)$ then assign x to category c_{h-1}.

What is the output?

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Lessons learned

- In order to aid decision making we need to handle preferences: learn, model and aggregate them.
- Preferences are ultimately binary relations. Numerical representations are useful, but not strictly necessary.
- "Weights" do not exist independently. They are not primitives, but second order models.
- There is no universal preference aggregation procedure and will never exist one. We always need to justify why we adopt that precise one and for which purpose.
- Providing decision aiding is not computing the output of a given procedure, but being able to explain, justify, use and revise this output.

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