Multi-attribute Value Theory

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Outline

Do numbers provide evidence?

alternatives	g_1	g_2
h	2000	500
a	160	435
b	400	370
c	640	305
d	880	240
e	1120	175
f	1360	110
g	1600	45

Table: Weighted sum

Do numbers provide evidence?

alternatives	g_1	g_2	g_1^n	g_2^n
h	2000	500	100.00	100.00
a	160	435	8.00	87.00
b	400	370	20.00	74.00
c	640	305	32.00	61.00
d	880	240	44.00	48.00
e	1120	175	56.00	35.00
f	1360	110	68.00	22.00
g	1600	45	80.00	9.00

Table: Weighted sum

Do numbers provide evidence?

alternatives	g_1	g_2	g_1^n	g_2^n	Score	Rank
h	2000	500	100.00	100.00	100.0	1
a	160	435	8.00	87.00	47.5	2
b	400	370	20.00	74.00	47.0	3
c	640	305	32.00	61.00	46.5	4
d	880	240	44.00	48.00	46.0	5
e	1120	175	56.00	35.00	45.5	6
f	1360	110	68.00	22.00	45.0	7
g	1600	45	80.00	9.00	44.5	8

Table: Weighted sum

Do numbers provide evidence?

alternatives	g_1	g_2	g_1^n	g_2^n	Score	Rank
h	2000	**700**	100.00	100.00		
a	160	435	8.00	62.14		
b	400	370	20.00	52.86		
c	640	305	32.00	43.57		
d	880	240	44.00	34.29		
e	1120	175	56.00	25.00		
f	1360	110	68.00	15.71		
g	1600	45	80.00	6.43		

Table: Weighted sum

Do numbers provide evidence?

alternatives	g_1	g_2	g_1^n	g_2^n	Score	Rank
h	2000	**700**	100.00	100.00	100.0	1
a	160	435	8.00	62.14	35.07	**8**
b	400	370	20.00	52.86	36.43	**7**
c	640	305	32.00	43.57	37.79	**6**
d	880	240	44.00	34.29	39.14	**5**
e	1120	175	56.00	25.00	40.50	**4**
f	1360	110	68.00	15.71	41.86	**3**
g	1600	45	80.00	6.43	43.21	**2**

Table: Weighted sum

Do numbers provide evidence?

alternatives	g_1	g_2	g_1^n	g_2^n	Score	Rank
h	2000	**700**	100.00	100.00	100.0	1
a	**165**	**450**	8.25	64.29	36.27	**8**
b	400	370	20.00	52.86	36.43	**7**
c	640	305	32.00	43.57	37.79	**6**
d	880	240	44.00	34.29	39.14	**5**
e	1120	175	56.00	25.00	40.50	**4**
f	1360	110	68.00	15.71	41.86	**3**
g	1600	45	80.00	6.43	43.21	**2**

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c	640	305	32.00	43.57	37.79	**6**
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e	1120	175	56.00	25.00	40.50	**4**
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g	1600	45	80.00	6.43	43.21	**2**

Table: Weighted sum

J.-Ch. Billaut, D. Bouyssou, Ph. Vincke, “Should you believe the Shanghai index?” *Scientometrics*, vol. 84, 237 - 263, 2010.

The Air Quality index

pollutant	CO_2	SO_2	O_3	dust
t_1	3	5	8	6

The Air Quality index

pollutant	CO_2	SO_2	O_3	dust
t_1	3	5	8	6
t_2	1	1	8	1

The Air Quality index

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t_2	1	1	8	1
t_3	7	7	7	7

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t_1	3	5	8	6
t_2	1	1	8	1
t_3	7	7	7	7

For the ATMO index t_3 is better than t_2.
If this index serves as an alert this is fine.
If this index serves to assess a policy this is counterintuitive.

Human Development Index

$$\mathrm{HDI} = \frac{\mathrm{LEI} + \mathrm{EAI} + \mathrm{GDPI}}{3}$$

$$\mathrm{LEI} = \frac{\text{life expectancy at birth} - 25}{85 - 25}$$

$$\mathrm{EAI} = \frac{2\mathrm{ALI} + \mathrm{ERI}}{3}$$

Human Development Index

$$\text{GDPI} = \frac{\text{transformed income} - W(100)}{W(40\,000) - W(100)}$$

where $W(x)$ represents the conversion of the GDP in standard monetary equivalents (USD) following Atkinson's formula.

Scale Normalisation

	life expectancy	EAI	GDPI
South Korea	71.5	.93	.97
Costa Rica	76.6	.86	.95

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	life expectancy	EAI	GDPI
South Korea	71.5	.93	.97
Costa Rica	76.6	.86	.95

If the scale is [85,25] then HDI(SK)>HDI(CR)
If the scale is [80,25] then HDI(CR)>HDI(SK)

Compensation

	life expectancy	ALI	ERI	real GDP	HDI
Gabon	54.1	.63	.60	3 641	.56
Solomon Islands	70.8	.62	.47	2 118	.58

Compensation

	life expectancy	ALI	ERI	real GDP	HDI
Gabon	54.1	.63	.60	3 641	.56
Solomon Islands	70.8	.62	.47	2 118	.58

A year of life is equivalent to 100.9 USD(equivalent).
If we transform this equivalent in real USD then poor's people life is less worth than rich people life!

Measurement

What is?

Measurement, in the broadest sense, is defined as the assignment of numerals to objects or events according to rules.

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More formal

From empirical evidence (ordered structures) to sets of numbers.

What do numbers represent?

If x is 50kg and y is 100kg, is y twice more heavy than x?

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If x is 50kg and y is 100kg, is y twice more heavy than x?

If x is $20^o C$ and y is $40^o C$, is y twice more hot than x?

Meaningfulness

- Information equivalent numerical scales
- Admissible transformations of numerical scales (create information equivalent representations)
- A class of admissible transformations univocally determines a scale type.

Measurement Scales

- Ordinal Scales
  (strictly increasing transformations)
- Interval Scales
  (positive affine transformations: $\varphi(x) = \alpha x + \beta$)
- Ratio Scales
  (positive homothetic transformations: $\varphi(x) = \alpha x$)
- Absolute Scales
  (identity transformations)

Is this sufficient?

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NO!!

Measures need to be useful.

Is this sufficient?

NO!!

Measures need to be useful.

x, y, z being the three dimensions of a solid

$x + y + z/3$ is the arithmetic mean, meaningful, but useless
xyz is the geometric mean, meaningful and useful

Is this sufficient?

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NO!!

Measures need to be legitimated.

Is this sufficient?

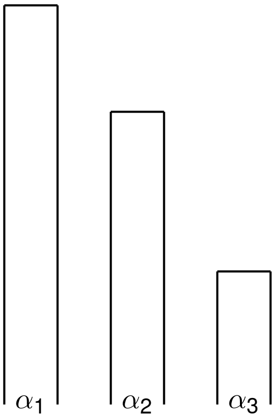
NO!!

Measures need to be legitimated.

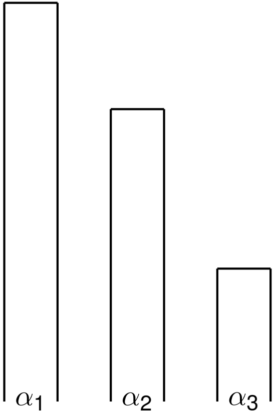
Racial statistics

are meaningful and (perhaps) useful, but in many places are not legitimated, if not forbidden.

Example



Example



$$\alpha_1 \succ \alpha_2 \succ \alpha_3$$

α_1	α_2	α_3
10	8	6
97	32	12
3	2	1

Any of the above could be a numerical representation of this empirical evidence.

Ordinal Scale: any increasing transformation of the numerical representation is compatible with the EE.

Further Example

Consider putting together objects and observing:

$$\alpha_1 \circ \alpha_5 > \alpha_3 \circ \alpha_4 > \alpha_1 \circ \alpha_2 > \alpha_5 > \alpha_4 > \alpha_3 > \alpha_2 > \alpha_1$$

Consider now the following numerical representations:

	L_1	L_2	L_3
α_1	14	10	14
α_2	15	91	16
α_3	20	92	17
α_4	21	93	18
α_5	28	99	29

L_1, L_2 and L_3 capture the simple order among α_{1-5}, but L_2 fails to capture the order among the combinations of objects.

Further Example

NB

For L_1 we get that $\alpha_2 \circ \alpha_3 \sim \alpha_1 \circ \alpha_4$
while for L_3 we get that $\alpha_2 \circ \alpha_3 > \alpha_1 \circ \alpha_4$.
We need to fix a "standard sequence".

Length

If we fix a "standard" length, a unit of measure, then all objects will be expressed as multiples of that unit.

Further Example

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For L_1 we get that $\alpha_2 \circ \alpha_3 \sim \alpha_1 \circ \alpha_4$
while for L_3 we get that $\alpha_2 \circ \alpha_3 > \alpha_1 \circ \alpha_4$.
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Length

If we fix a "standard" length, a unit of measure, then all objects will be expressed as multiples of that unit.

More complicated

Consider a Multi-attribute space:

$$X = X_1 \times \cdot X_n$$

to each attribute we associate an ordered set of values:

$$X_j = \langle x_j^1 \cdots x_j^m \rangle$$

An object x will thus be a vector:

$$x = \langle x_1^l \cdots x_n^k \rangle$$

Generally speaking ...

$$x \succeq y$$

$$\iff$$

$$\langle x_1^l \cdots x_n^k \rangle \succeq \langle y_1^i \cdots y_n^j \rangle$$

$$\iff$$

$$\Phi(f(x_1^l \cdots x_n^k), f(y_1^i \cdots y_n^j)) \geq 0$$

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	$70+\delta_1$	C	500	1500

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	$70+\delta_1$	C	500	1500

For what value of δ_1 a and a_1 are indifferent?

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	80	C	500	1500

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	80	C	500	1500
a_2	25	80	C	700	$1500+\delta_2$

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	80	C	500	1500
a_2	25	80	C	700	$1500+\delta_2$

For what value of δ_2 a_1 and a_2 are indifferent?

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	80	C	500	1500
a_2	25	80	C	700	1700

What that means?

	Commuting Time	Clients Exposure	Services	Size	Costs
a	20	70	C	500	1500
a_1	25	80	C	500	1500
a_2	25	80	C	700	1700

The trade-offs introduced with δ_1 and δ_2 allow to get $a \sim a_1 \sim a_2$

Claims

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We can be rigourous without being quantitative. The key concept is meaningfulness.

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Measuring is most of the times a decision aiding activity where decision aiding methodology applies.

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Claim 2

Measuring is most of the times a decision aiding activity where decision aiding methodology applies.

Claim 3

Decisions are not in the date, but in the values. Data are necessary, but not sufficient.

What is evaluation?

Measuring values (of decision makers, voters, customers ...)

What is the empirical evidence for value measurement?

- revealed preferences from customers' behaviour in markets
- subjective preferences from direct or indirect observation

Can we measure subjective preferences?

Yes!!

Empirical evidence: preference statements

- direct approach: indifference swaps;
- indirect approach: value estimation through learning algorithms.

How do we measure better?

Let's go more formal.

- Let $x, y, z \ldots$ be competing projects within set A;
- Let $d_j(x)$ representing the consequences of project x on dimension d_j;
- Let $d_j(A)$ representing the set of all consequences for all projects in A.

The first step consists in verifying that:

$$\forall j \in D \;\; \exists \succeq_j \subseteq d_j(A)^2$$

such that $\succeq_j$ is a weak order (consequences should be completely and transitively ordered).

How do we measure better?

If the previous hypothesis is verified then

$$\forall j \in D \;\; \exists h_j : A \mapsto \mathbb{R} : d_j(x) \succeq d_j(y) \Leftrightarrow h_j(x) \geq h_j(y)$$

In other terms for each dimension we can establish a real valued function respecting the decision maker's preferences.

This function is ONLY an ordinal measure of the preferences

Example-1

Suppose you have 4 projects x, y, z, w of urban rehabilitation and an assessment dimension named “esthetics”. You have:
- $d_e(x) =$ statue;
- $d_e(y) =$ fountain;
- $d_e(z) =$ garden;
- $d_e(w) =$ kid’s area;

Preferences expressed could be for instance:
$d_e(x) \succ d_e(y) \succ d_e(z) \sim d_e(w)$
A possible numerical representation could thus be:
$h_e(x) = 3,\ h_e(y) = 2,\ h_e(z) = h_e(w) = 1$

Example-2

Suppose you have 4 projects x, y, z, w of urban rehabilitation and an assessment dimension named “land use”. You have:
- $d_l(x) = 100$sqm;
- $d_l(y) = 50$sqm;
- $d_l(z) = 1000$sqm;
- $d_l(w) = 500$sqm;

Preferences expressed could be for instance (suppose the decision maker dislikes land use:
$d_e(y) \succ d_e(x) \succ d_e(w) \sim d_e(z)$
A possible numerical representation could thus be:
$h_e(y) = 4$, $h_e(x) = 3$, $h_e(w) = 2$ $h_e(z) = 1$, but also:
$h_e(y) = 50$, $h_e(x) = 100$, $h_e(w) = 500$ $h_e(z) = 1000$

Is this sufficient?

For the time being we have the following table:

	d_1-h_1	d_2-h_2	$\ldots$	d_n-h_n
x				
y				
z				
w				
$\vdots$				

The consequences of each action and the numerical representation of the decision maker's preferences (ordinal).

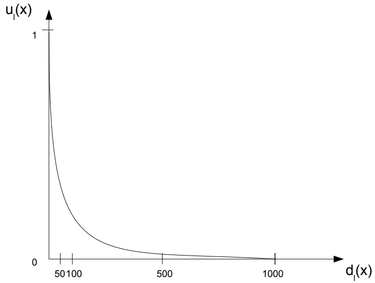
Is this sufficient?

NO!

We need something more rich. We need to know, when we compare x to y (and we prefer x) if this preference is "stronger" to the one expressed when comparing (on the same dimension) z fo w.

We need to compare differences of preferences

An example



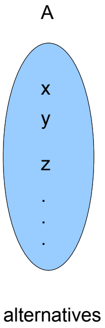
For instance, if the above function represents the value of “land use” it is clear that the difference between 50sqm and 100sqm is far more important from the one between 500sqm and 1000sqm.

First Summary

Let's summarise our process until now.

- We get the alternatives.
- We identify their consequences for all relevant dimensions.
- These consequences are ordered for each dimension using the decision maker's preferences.
- We compute the value function measuring the differences of preferences (for each dimension).

First Summary



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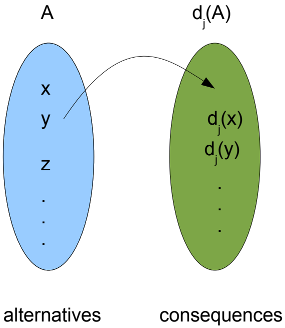

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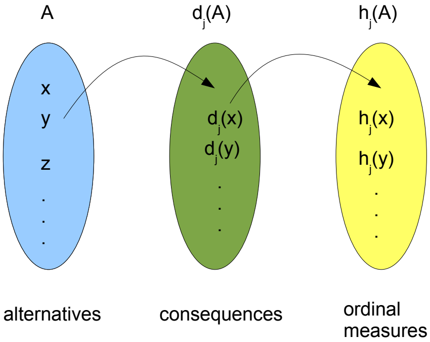

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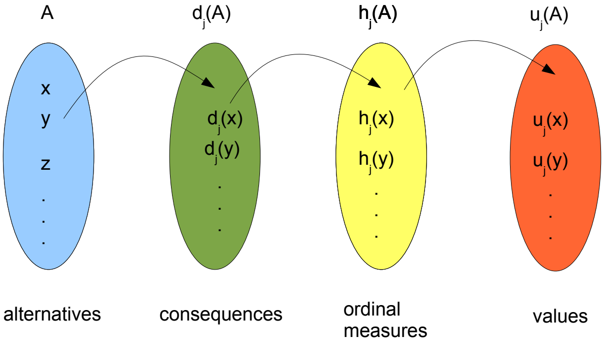

Is all that sufficient?

NO!

1. The problem is that we need to be able to compare the differences of preferences on one dimension to the differences of preferences on another one (let's say differences of preferences on land use with differences of preferences on esthetics.
2. At the same time we need to take into account the intuitive idea that for a given decision maker certain dimensions are more "important" than other ones.

Principal Hypotheses

1. The different dimensions are separable.
2. Preferences on each dimension are independent.
3. Preferences on each dimension are measurable in terms of differences.
4. Good values on one dimension can compensate bad values on another dimension.

Principal Hypotheses

Under the previous hypotheses we can construct a global value function $U(x)$ as follows:

$$U(x) = \sum_j u_j(x)$$

and in case we use normalised (in the interval [0,1]) marginal value functions $\bar{u}_j$ then:

$$U(x) = \sum_j w_j \bar{u}_j(x)$$

Principal Hypotheses

where: w_j should represent the importance of the marginal functions;
If $h_j(x)$ represent the ordinal values of dimension j then $u_j(d_j(\underline{x})) = 0$ where $d_j(\underline{x})$ is the worst value of h_j
and in case we use normalised value functions then $u_j(d_j(\overline{x})) = 1$ where $d_j(\overline{x})$ is the best value of h_j.

Standard Protocol

1. Fix arbitrary one dimension as the reference for which the value function will be linear (there is no loss of generality doing so).
2. Fix a number of units diving entirely the reference value function, thus fixing the unit of value U_1.
3. Une indifference questions (see later) in order to find equivalent values for the other dimensions.
4. The segments between the equivalent values will shape the other value functions.
5. The ratio of units used to describe each value function with respect to the units for the reference one establishes the trade-offs among the dimensions.

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Indifference Questions

Given d_r as the reference dimension, h_r being the ordinal preferences we want to establish a value function for dimension d_k. Consider a fictitious object x for which we have $\langle h_r(x), h_k(x) \rangle$. The key question is:

$$\langle h_r(x), h_k(x) \rangle \sim \langle h_r(\bar{x}), ? \rangle$$

What should be the measure on dimension k of an object $\bar{x}$ whose measure on the reference dimension r is such that the $u_r(\bar{x}) = u_r(x) + U_1$ if x and $\bar{x}$ should be indifferent for the decision maker?

Indifference Questions

Once you get the answer $h_k(\bar{x})$ from the decision maker you go ahead:

$$\langle h_r(x), h_k(\bar{x})\rangle \sim \langle h_r(\bar{x}), ?\rangle \quad \rightarrow h_k(\bar{\bar{x}})$$

$$\langle h_r(x), h_k(\bar{\bar{x}})\rangle \sim \langle h_r(\bar{x}), ?\rangle \quad \rightarrow h_k(\bar{\bar{\bar{x}}})$$

Until the whole set of measures of dimension k has been used.

TIPS

- TIP1 Start considering a point x at the middle of both scales h_r and h_k.
- TIP2 Then start deteriorating on the reference dimension by one unit of value at time (thus the dimension under construction has to improve) until the upper scale of h_k is exhausted.
- TIP2 Then start improving on the reference dimension by one unit of value at time (thus the dimension under construction has to deteriorate) until the lower scale of h_k is exhausted.

What do we get?

We have $U(x) = u_r(x) + u_k(x)$ by definition.
We also have $U(\bar{x}) = u_r(\bar{x}) + u_k(\bar{x})$ after questioning.
And since x and $\bar{x}$ are considered indifferent $U(x) = U(\bar{x})$.
Then we get $u_r(x) + u_k(x) = u_r(x) + U_1 + u_k(\bar{x})$ by construction.
We obtain $u_k(\bar{x}) = u_k(x) - U_1$.

Going ahead recursively we found the point $\underline{x}$ at the bottom of the scale for which by definition $u_k(\underline{x}) = 0$ (by definition). Using linear segments between all the points discovered we shape the value function u_k.

Example

You have to choose among competitive projects assessed against 3 attributes: cost, esthetics and mass. As far as the cost is concerned the scale goes from 5M€ to 10M€. Esthetics are assessed on a subjective scale going from 0 to 8. Mass is measured in kg and the scale goes from 1kg to 5kg. In this precise moment you have under evaluation the following four ones:

project	c	e	m
A	6,5M€	3	3kg
B	7,5M€	4	4,5kg
C	8M€	6	2kg
D	9M€	7	1,5kg

Which is the “best choice”?

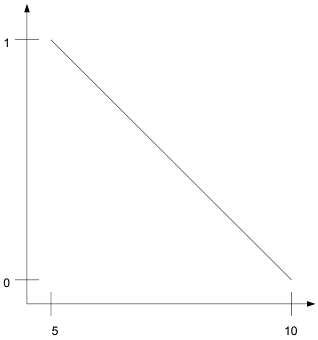
Preferences

First we need to establish appropriate preferences. Suppose in your case the following ones:

- you prefer the less expensive to the more expensive (cost);
- you prefer “pretty” to “less pretty” (esthetics);
- you prefer “heavy” to “less heavy” (mass).

Cost Value Function

Without loss of generality we establish the cost as reference criterion with a linear value function such that $u_c(5M€) = 1$ and $u_c(10M€) = 0$. We fix the value unit $U_1 = 0{,}5M€$.



Cost Value Function

Esthetics Value Function

In order to construct the value function of Esthetics we proceed with the following dialog:

$$\langle 7.5\text{M€}, 4\rangle \sim \langle 8\text{M€}, ?\rangle$$

Consider a project which costs 7.5€ and is assessed on esthetics with 4, and a project which costs 8M€ (one unit of value less in this case), how much should the second project be improved in esthetics in order to be indifferent to the first one?
Suppose we get an answer of 5: $\langle 7.5\text{M€}, 4\rangle \sim \langle 8\text{M€}, 5\rangle$
We repeat now the question using the new value:

$$\langle 7.5\text{M€}, 5\rangle \sim \langle 8\text{M€}, ?\rangle$$

We now get an answer of 6.

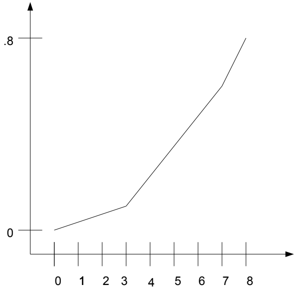
Esthetics Indifferences

We can summarise the dialog as follows:

$$\langle 7.5M€, 4\rangle \sim \langle 8M€, 5\rangle$$
$$\langle 7.5M€, 5\rangle \sim \langle 8M€, 6\rangle$$
$$\langle 7.5M€, 6\rangle \sim \langle 8M€, 7\rangle$$
$$\langle 7.5M€, 7\rangle \sim \langle 8M€, 7.5\rangle$$
$$\langle 7.5M€, 7.5\rangle \sim \langle 8M€, 8\rangle$$
$$\langle 7.5M€, 4\rangle \sim \langle 7M€, 3\rangle$$
$$\langle 7.5M€, 3\rangle \sim \langle 7M€, 1.5\rangle$$
$$\langle 7.5M€, 1.5\rangle \sim \langle 7M€, 0\rangle$$

Esthetics Value Function

The previous dialog will result in the following value function.



Esthetics Value Function

Mass Value Function

In order to construct the value function of Mass we proceed with the following dialog:

$$\langle 7.5M€, 3.1\rangle \sim \langle 8M€, ?\rangle$$

Consider a project which costs 7.5€ and weighs 3.1kg and a project which costs 8M€ (one unit of value less in this case), how much should the second project be improved in mass in order to be indifferent to the first one? Suppose we get an answer of 3.5kg: $\langle 7.5M€, 3.1\rangle \sim \langle 8M€, 3.5\rangle$
We repeat now the question using the new value:

$$\langle 7.5M€, 5\rangle \sim \langle 8M€, ?\rangle$$

We now get an answer of 3.9.

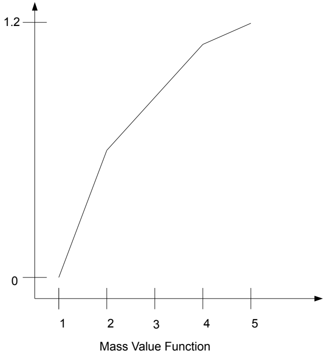
Mass Indifferences

We can summarise the dialog as follows:

$$\langle 7.5M€, 3.1\rangle \sim \langle 8M€, 3.5\rangle$$
$$\langle 7.5M€, 3.5\rangle \sim \langle 8M€, 3.9\rangle$$
$$\langle 7.5M€, 3.9\rangle \sim \langle 8M€, 5\rangle$$
$$\langle 7.5M€, 3.1\rangle \sim \langle 7M€, 2.7\rangle$$
$$\langle 7.5M€, 2.7\rangle \sim \langle 7M€, 2.3\rangle$$
$$\langle 7.5M€, 2.3\rangle \sim \langle 7M€, 1.9\rangle$$
$$\langle 7.5M€, 1.9\rangle \sim \langle 7M€, 1.75\rangle$$
$$\langle 7.5M€, 1.75\rangle \sim \langle 7M€, 1.6\rangle$$
$$\langle 7.5M€, 1.6\rangle \sim \langle 7M€, 1.45\rangle$$
$$\langle 7.5M€, 1.45\rangle \sim \langle 7M€, 1.3\rangle$$
$$\langle 7.5M€, 1.3\rangle \sim \langle 7M€, 1.15\rangle$$
$$\langle 7.5M€, 1.15\rangle \sim \langle 7M€, 1\rangle$$

Mass Value Function

The previous dialog will result in the following value function.



Final calculations

Having obtained the three value functions we can now calculate the values of the four projects for each of them.

$$
\begin{array}{lll}
u_c(A) = 0.7 & u_e(A) = 0.2 & u_m(A) = 0.875 \\
u_c(B) = 0.5 & u_e(B) = 0.3 & u_m(B) = 1.160 \\
u_c(C) = 0.4 & u_e(C) = 0.5 & u_m(C) = 0.625 \\
u_c(D) = 0.2 & u_e(D) = 0.6 & u_m(D) = 0.330
\end{array}
$$

Final Results

Finally we get

$$\begin{aligned} U_c(A) &= 0.7 + 0.2 + 0.875 = 1.775 \\ U_c(B) &= 0.5 + 0.3 + 1.160 = 1.960 \\ U_c(C) &= 0.4 + 0.5 + 0.625 = 1.525 \\ U_c(D) &= 0.2 + 0.6 + 0.330 = 1.130 \end{aligned}$$

The project which maximises the decision maker's value is B.

Where did the weight disappear?

NOWHERE

Suppose we were using normalised value functions which have to be “weighted”. We recall that in such a case we have:

$$U(x) = \sum_j w_j \bar{u}_j(x)$$

Consider the first indifference sentence about esthetics. We had: $\langle 7.5\text{M€}, 4\rangle \sim \langle 8\text{M€}, 5\rangle$. We get:
$w_c\bar{u}_c(7.5\text{M€}) + w_e\bar{u}_e(4) = w_c\bar{u}_c(8\text{M€}) + w_e\bar{u}_e(5)$
where:
- w_c and w_e represent the “weights” of cost and esthetics respectively;
- and $\bar{u}_c$ and $\bar{u}_e$ are the normalised value functions.

Here are the weights ...

By construction $u_c(x) = \bar{u}_c(x)$. We get:
$w_c(\bar{u}_c(7.5M€) - \bar{u}_c(8M€)) = w_e(\bar{u}_e(5) - \bar{u}_e(4))$. Thus:

$$\frac{w_e}{w_c} = \frac{\bar{u}_c(7.5M€) - \bar{u}_c(8M€)}{\bar{u}_e(5) - \bar{u}_e(4)}$$

However, $\bar{u}_c(7.5M€) - \bar{u}_c(8M€) = 1/10$ of the cost value function (by construction) and $\bar{u}_e(5) - \bar{u}_e(4) = 1/8$ of the esthetics value function as it results from the dialog. Using the same procedure for mass we get:
- $w_e/w_c = 0.8$ meaning that esthetics represents 80% of the cost value (this is the esthetics trade-off);
- $w_m/w_c = 1.2$ meaning that mass represents 120% of the cost value (this is the mass trade-off);

Conclusion and tips

Tip1 Not surprisingly the "weight" of each criterion is represented by the maximum value it attains.

Tip2 It is better not to use any "weights" when constructing value functions, since it can generate confusion to the decision maker. We can explain the relative importance of each criterion using the trade-offs.

So called "weights" are the trade-offs among the value functions and as such are established as soon as the value functions are constructed. They do not exist independently and is not correct to ask the decision maker to express them.

Uncertainty

	θ_1	$\cdots \theta_i$	θ_m
α_1			
$\vdots$			
α_j		$\theta_i(\alpha_j)$	
α_n			

where:
- α_j are potential actions;
- θ_i are possible scenarios;
- $\theta_i(\alpha_j)$ are the consequences

Utility Functions

If

$$v(x) = \sum_{i=1}^{m} p(\theta_i)u(\theta_i(x))$$

Utility Functions

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$u(\theta_i(x))$ are just value functions for scenario i (instead for criterion i) and $p(\theta_i)$ is the importance of scenario θ_i (thus the probability).

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Hence

Probability does not exist. Probability is not a primitive.

Multi-attribute Utility

A two steps value measurement

either aggregate first the criteria and then the scenarios
or aggregate first the scenarios and then the criteria

Why not linear?

Preferential independence is rare and different to demonstrate

- prospect theory and other behavioural approaches to utility
- k-additive value functions (taking into account interactions among k criteria
- moltiplicative (and more) value functions
- CP nets, GAI networks and the similar

Why not linear?

- Non linear models are more accurate and most of the times will fit better client's preferences.
- However, is MUCH more expensive to learn and construct non linear value functions.
- General framework for an axiomatic study: conjoint measurement theory.

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