ELECTRE METHODS (PART I)

José Rui FIGUEIRA
(figueira@tecnico.ulisboa.pt)

Universidade de Lisboa

MCDM Summer School, Chania-Crete, Greece

Contents

Main references for this talk:

1. Figueira, J., B. Roy, and V. Mousseau (2005). ELECTRE methods. In J. Figueira, S. Greco, and M. Ehrgott (Eds.), *Multiple Criteria Decision Analysis: State of the Art Surveys*, pp. 133162. New York, U.S.A.: Springer Science + Business Media, Inc.

2. Figueira, J.R., S. Greco, B. Roy, and R. Słowiński, (2010). ELECTRE methods: Main features and recent developments. In C. Zopounidis and P. Pardalos (Eds.), *Handbook of Multicriteria Analysis*, Chapter 4, New York, USA: Springer.

3. Greco, S., R. Słowiński, J.R. Figueira, and V. Mousseau (2010). Robust ordinal regression. In M. Ehrgott, J.R. Figueira, and S. Greco (Eds.), *Trends in Multiple Criteria Decision Analysis*, pp. 273–320. New York, U.S.A.: Springer Science + Business Media, Inc.

A decision aiding situation

1. Imagine that in a company or institution, a CEO is confronted with a certain decision aiding situation and has to make a decision.
2. The CEO needs the help of an analyst (an in-house operational service, a consultant, or a university research team).
3. Two key elements in a decision aiding situation are: The Analyst and the Decision Maker (DM). The latter is here represented by the CEO.

The decision aiding activity is based on three fundamental pillars:

1. The actions (formal definition of the possible actions or alternatives).
2. The consequences (aspects, attributes, characteristics, . . . of the actions that allow to compare them).
3. The modeling of a preference system (it consists of an implicit or explicit process, that for each pair of actions envisioned, assigns one and only one of the three possibilities: indifference, preference, or incomparability).

Based on the above three pillars:

1. The analyst should try to obtain a coherent structured set of results in order to guide the decision aiding process and facilitate the communications about the decisions.
2. The analyst must follow an approach that leads or aims to produce knowledge from a certain number working hypotheses defined *a priori*.
3. This approach should be based on models that are, at least co-constructed interactively with the DM.

Based on the above three pillars:

4. During the co-construction process, that takes into account the values of the DM, contradictory judgements or ambiguities may occur.

5. The analyst must admit that the novelty of these questions can bring (the DM) or the person this questioned to revise certain pre-existing preferences momentarily and locally.

Basic data

1. $A = \{a_1, a_2, \ldots, a_i, \ldots, a_m\}$ is the set of m potential actions. This set can be partially known *a priori* (it is frequent in sorting problems).

2. $F = \{g_1, g_2, \ldots, g_j, \ldots, g_n\}$ is a coherent family of criteria, with $n \geqslant 3$.

3. $g_j(a_i)$ is the performance of action a_i on criterion g_j, for all $a_i \in A$ and $g_j \in F$. A performance table M can thus be built.

4. Assume w.l.g. that the higher the performance $g_j(a)$ is, the better for the DM (increasing direction of preference).

Four main comprehensive preference situations

1. I (Indifference)
2. P (strict preference)
3. Q (hesitation : weak preference)
4. R (incomparability).

(For more details see Figueira et al., 2010)

Pseudo-criterion

A pseudo-criterion is a function g_j associated with two threshold functions, $q_j(\cdot)$ and $p_j(\cdot)$, satisfying the following condition: for all ordered pairs of actions $(a, a') \in A \times A$ such that $g_j(a) \geqslant g_j(a')$, $g_j(a) + p_j(g_j(a'))$ and $g_j(a) + q_j(g_j(a'))$ are non-decreasing monotone functions of $g_j(a')$, such that $p_j(g_j(a')) \geqslant q_j(g_j(a')) \geqslant 0$, for all $a \in A$.

Partial binary relations (1)

1. $g_j(a) - g_j(a') > p_j(g_j(a')) \Leftrightarrow aP_ja'$,
2. $q_j(g_j(a')) < g_j(a) - g_j(a') \leqslant p_j(g_j(a')) \Leftrightarrow aQ_ja'$,
3. $-q_j(g_j(a)) \leqslant g_j(a) - g_j(a') \leqslant q_j(g_j(a')) \Leftrightarrow aI_ja'$.

Partial binary relations (2)

1. $S_j = P_j \cup Q_j \cup I_j$
2. aS_ja' means that "a is at least as good as a'" on criterion g_j.
3. When aS_ja' the voting power of criterion g_j, denoted by w_j is taken in total (assume w.l.g. that $w_1 + w_2 + \ldots + w_n = 1$).

Let $S = P \cup Q \cup I$, whose meaning "is at least as good as".

Comprehensive outranking

Consider two actions, a and a' and the relation $\succ = P \cup Q$. Four situations may occur:

1. aSa' and not$(a'Sa)$, i.e., $a \succ a'$ (a is preferred in a broader sense to a').
2. $a'Sa$ and not(aSa'), i.e., $a' \succ a$ (a' is preferred in a broader sense to a).
3. aSa' and $a'Sa$, i.e., aIa' (a is indifferent to a').
4. not(aSa') and not$(a'Sa)$, i.e., aRa' (a is incomparable to a').

Concordance

1. Concordance. To validate aSa', a sufficient majority of criteria in favor of this assertion must occur.

2. The comprehensive concordance index $c(a,a')$ for each pair of actions $(a,a') \in A \times A$, for all $g_j \in F$ is fundamental to all the ELECTRE methods in order to compute a concordance matrix C.

$$c(a,a') = \sum_{\{j \mid g_j \in C(a\{P,Q,I\}a')\}} w_j + \sum_{\{j \mid g_j \in C(a'Qa)\}} w_j \varphi_j$$

where

$$\varphi_j = \frac{g_j(a) - g_j(a') + p_j}{p_j - q_j}$$

Voting power

This index comprises the summation of the voting power of the criteria that clearly are in favor of the assertion aSa', plus the summation of the fraction, φ_j, of the voting power for those criteria included in the hesitation group.

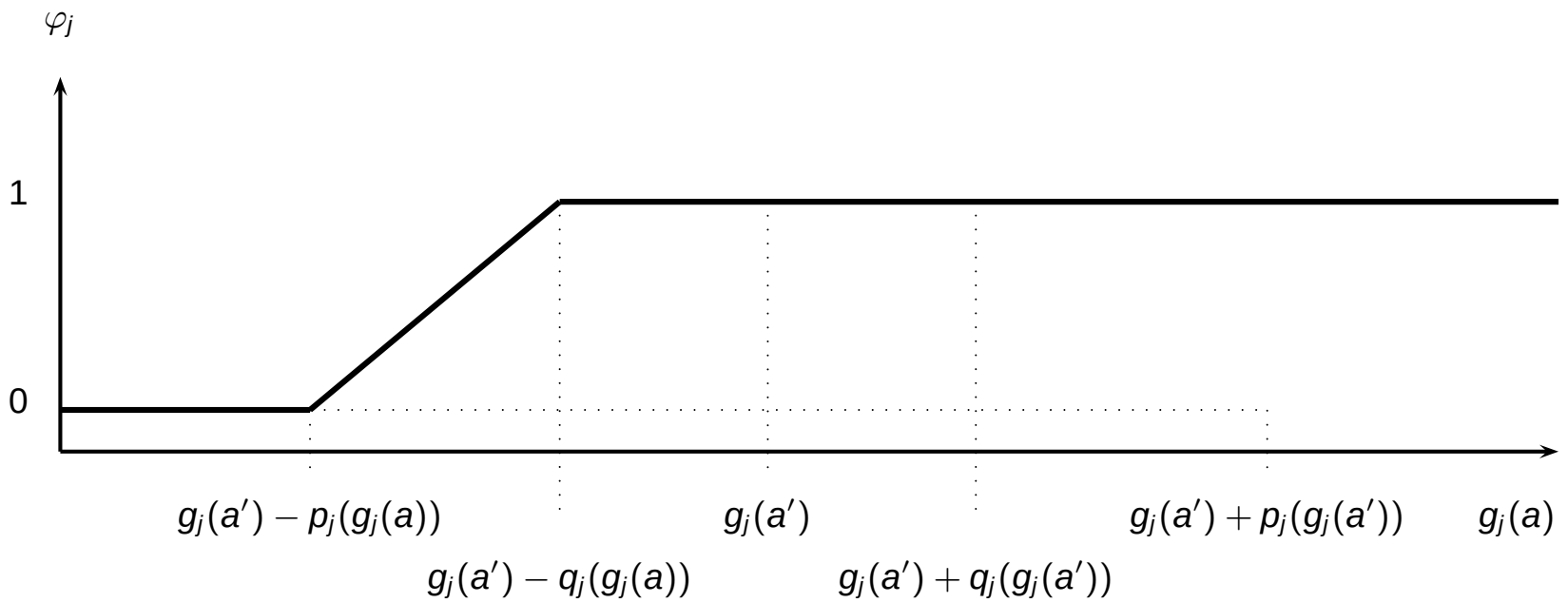


Figure: Variation of φ_j for a given $g_j(a')$ and variable $g_j(a)$

Discordance

1. Discordance. The assertion aSa' cannot be validated if a minority of criteria is strongly against this assertions.
2. The concept of veto threshold, v_j, gives the possibility to the criterion g_j to impose its veto power. It means that $g_j(a')$ is so much better than $g_j(a)$, that is not possible to allow that aSa'
3. The computation of the partial discordance indices leads to the construction of a discordance matrix, D.
4. The application of both types of indices is related to a specific ELECTRE method. For example, in ELECTRE TRI they are "combined" with $c(a, a')$ to define a degree of credibility of the assertion aSa' (fuzzy relation).

Partial discordance index

$$d_j(a,a') = \begin{cases} 1 & \text{if } g_j(a) - g_j(a') < -v_j(g_j(a)), \\ \frac{g_j(a) - g_j(a') + p_j(g_j(a))}{p_j(g_j(a)) - v_j(g_j(a))} & \text{if } -v_j(g_j(a)) \leqslant g_j(a) - g_j(a') < -p_j(g_j(a)), \\ 0 & \text{if } g_j(a) - g_j(a') \geqslant -p_j(g_j(a)). \end{cases}$$

Reminder and additional notation

1. We use k_j as the non-normalized weights for each criterion)
   - $C(aSa')$ is the coalition of criteria in favor of the assertion aSa'.
   - $W\{C(aSa')\} = \sum_{\{j\,:\,g_j \in C(aSa')\}} w_j$ is the weight or power of the coalition $C(aSa')$.
2. $q_j(\cdot)$ is the indifference threshold of criterion g_j.
3. $p_j(\cdot)$ is the preference threshold of criterion g_j.
4. $v_j(\cdot)$ is a veto threshold of criterion g_j.

Location of a new hotel

1. Table below presents the performances of the five sites - a, b, c, d, and e - according to the five criteria.
2. The performances of criterion g_1 (investment costs) are expressed in thousands of €, designated K€.
3. The indifference and the preference thresholds assigned to this criterion are $q_1(g_1(x)) = 500 + 0.03g_1(x)$ K€ and $p_1(g_1(x)) = 1000 + 0.05g_1(x)$ K€, respectively, where x is the worst of the two actions.
4. The performances of criterion g_2 (annual costs) are also expressed in K€; the thresholds assigned to this criterion are $q_2(g_1(x)) = 50 + 0.05g_1(x)$ K€ and $p_2(g_1(x)) = 100 + 0.07g_1(x)$ K€, respectively.

Location of a new hotel

1. The performances of criteria g_3 (recruitment), g_4 (image), and g_5 (access) are expressed on the following seven-level qualitative scale: very bad (1), bad (2), rather bad (3), average (5), rather good (5), good (6), and very good (7). The values between parenthesis can be used in ELECTRE methods to code the different verbal statements.

2. Other ways of coding the verbal scale through the use of numerical values could be used by adjusting the thresholds values (see Martel and Roy, 2006).

3. The indifference threshold for each criterion has been set at one on the seven-level scale and the preference threshold at two levels.

4. In this example there is no veto.

Performances Table

1. Quantitative criteria: g_1 (investment costs) and g_2 (annual costs)
2. Qualitative criteria: g_3 (recruitment), g_4 (image), and g_5 (access)

	g_1[min]	g_2[min]	g_3[max]	g_4[max]	g_5[max]
a	13 000 K€	3 000 K€	Average	Average	Average
b	15 000 K€	2 500 K€	Good	Bad	Very Good
c	10 900 K€	3 400 K€	Good	Good	Very Bad
d	15 500 K€	3 500 K€	Good	Good	Good
e	15 000 K€	2 600 K€	Good	Very Bad	Bad
k_j	5	4	3	3	3

Pairwise comparison

1. Does a outrank d, aSd? For the moment we cannot answer this question.
2. The coalition of criteria in favor of aSd: $C(aSd) = \{g_1, g_2\}$
3. The power of this coalition: $W\{C(aSd)\} = \frac{4+5}{18} = 0.5$ (normalized)
4. What about dSa?

Each ELECTRE method comprises two main procedures:

Two procedures

1. The first procedure is a Multiple Criteria Aggregation Procedure (MCAP) that builds one or possibly several outranking relations aim to compare, in a comprehensive way, each ordered pair of actions.

2. The second procedure, called Exploitation Procedure (EP) is used to obtain adequate results from which recommendations can be derived.

3. The nature of the results depends of the specific *problematique*.

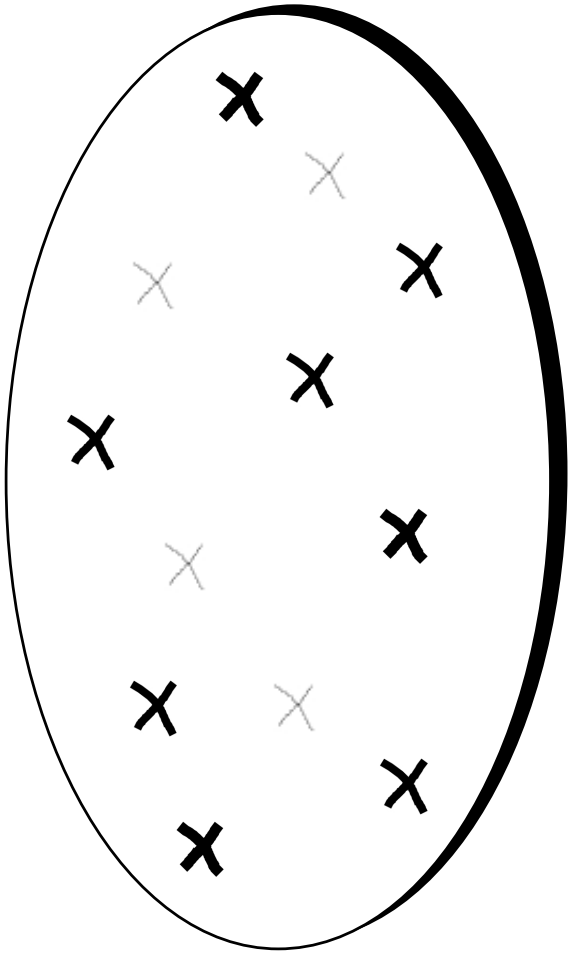
MCAP of ELECTRE III

It is modeled through a credibility index i.e. a fuzzy measure denoted by $\sigma(a,a') \in [0,1]$, which combines $c(a,a')$ and $d_j(a,a')$:

$$\sigma(a,a') = c(a,a') \prod_{j \in \mathcal{J}(a,a')} \frac{1-d_j(a,a')}{1-c(a,a')},$$

where $j \in \mathcal{J}(a,a')$ if and only if $d_j(a,a') \geqslant c(a,a')$.

Choosing: Selecting a restricted number as small as possible of potential actions, which justify to eliminating others.



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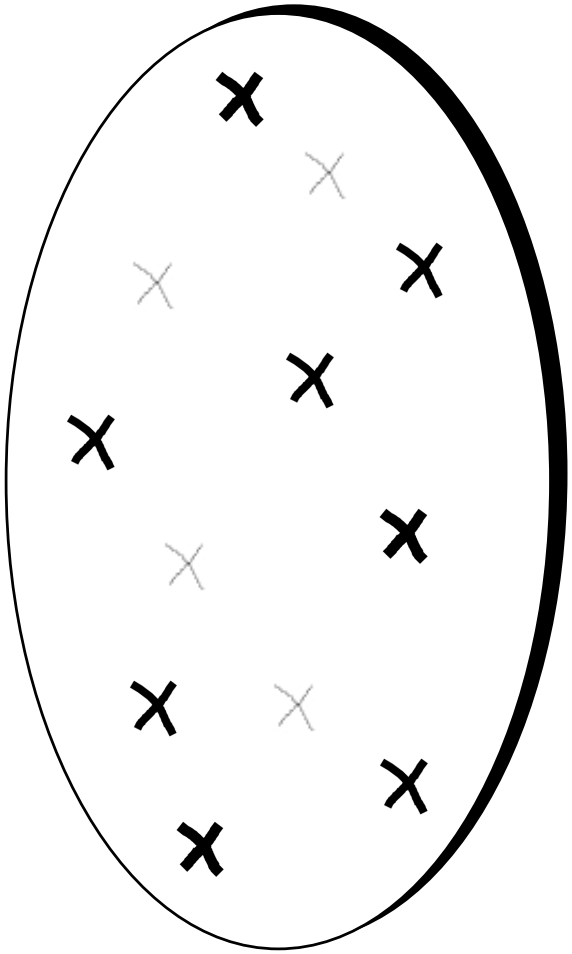

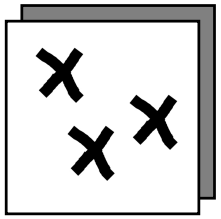

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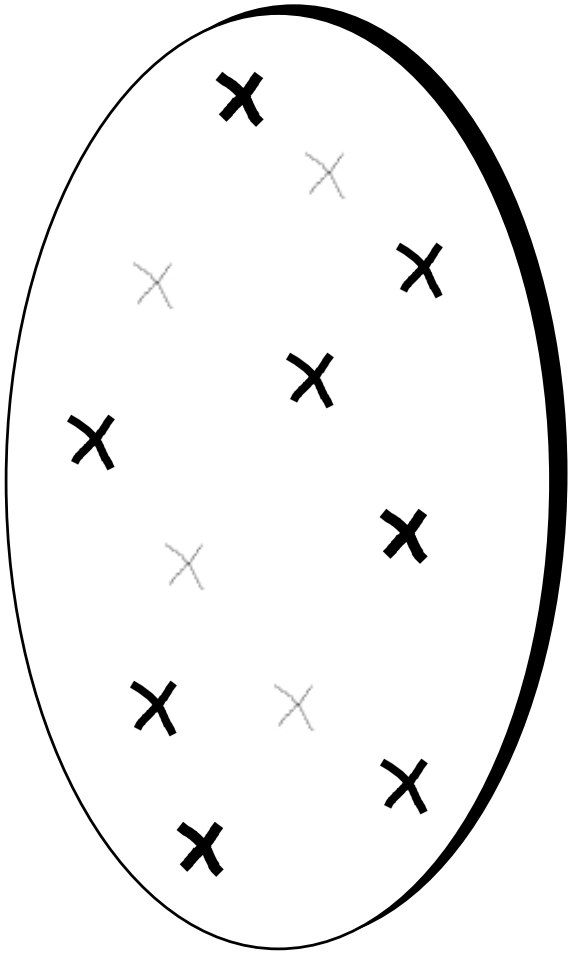
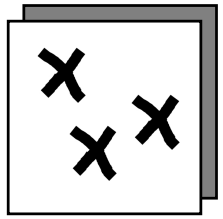
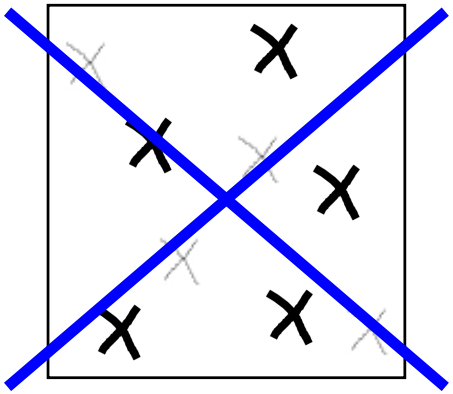


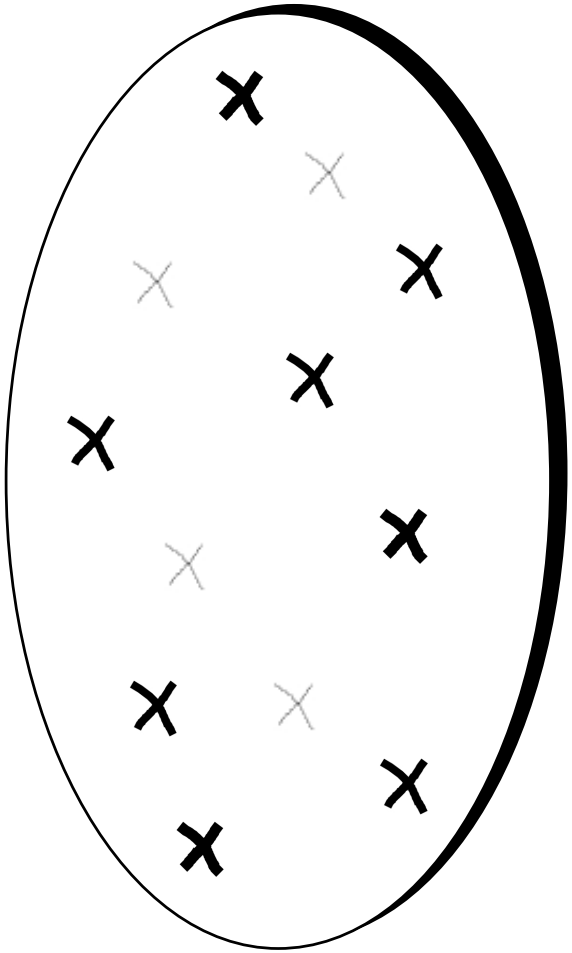
Choice set





Actions rejected

Ranking: Ranking of actions from the best to the worst, with the of ties (*ex aequo*) and incomparabilities.



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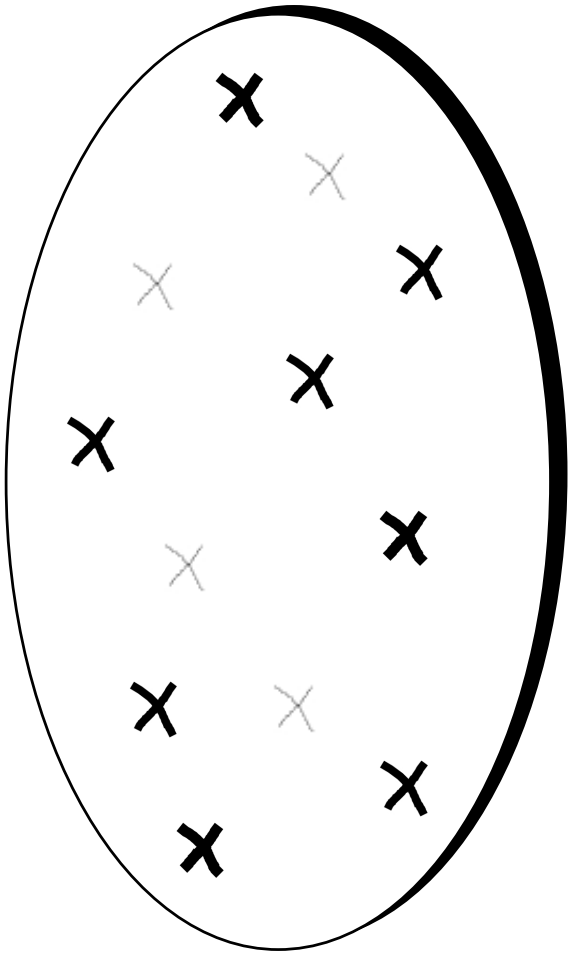
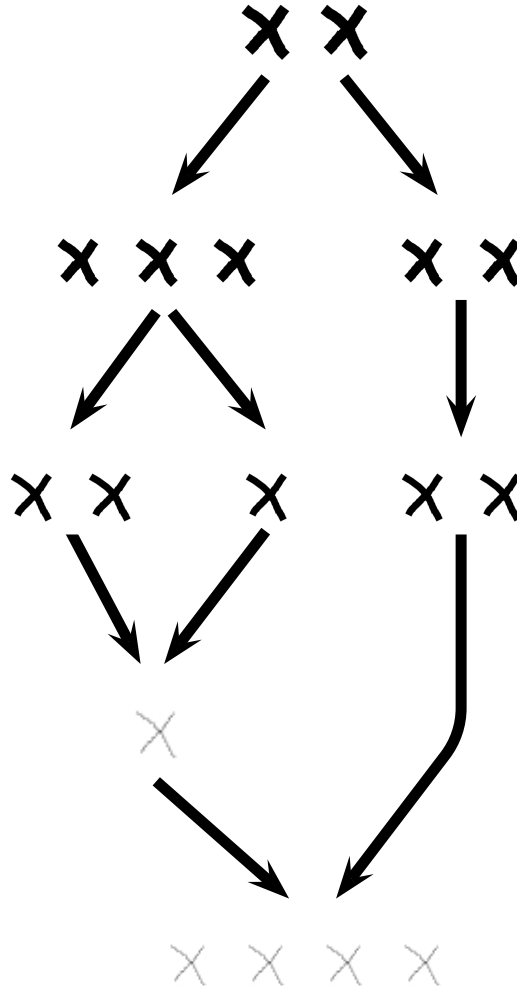
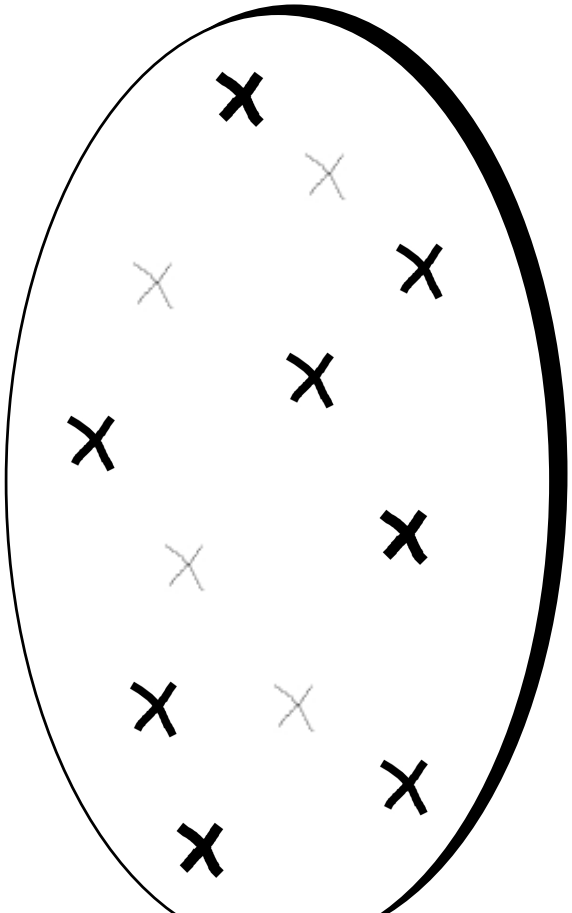




Ordinal classification or sorting: Assigning each potential action to one of the categories among those of a family previously defined; the categories are ordered, in general, from the worst to the best one.



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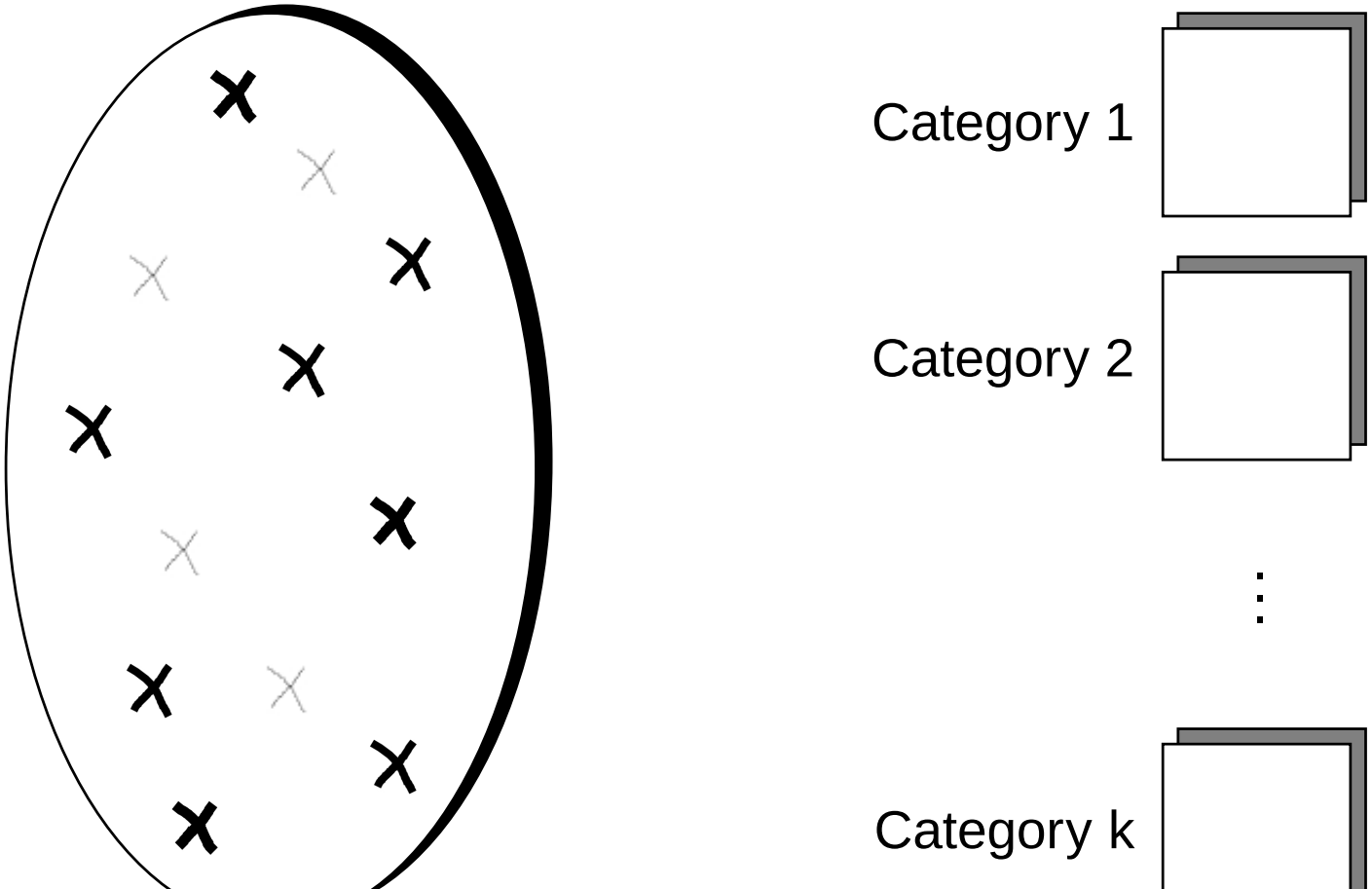

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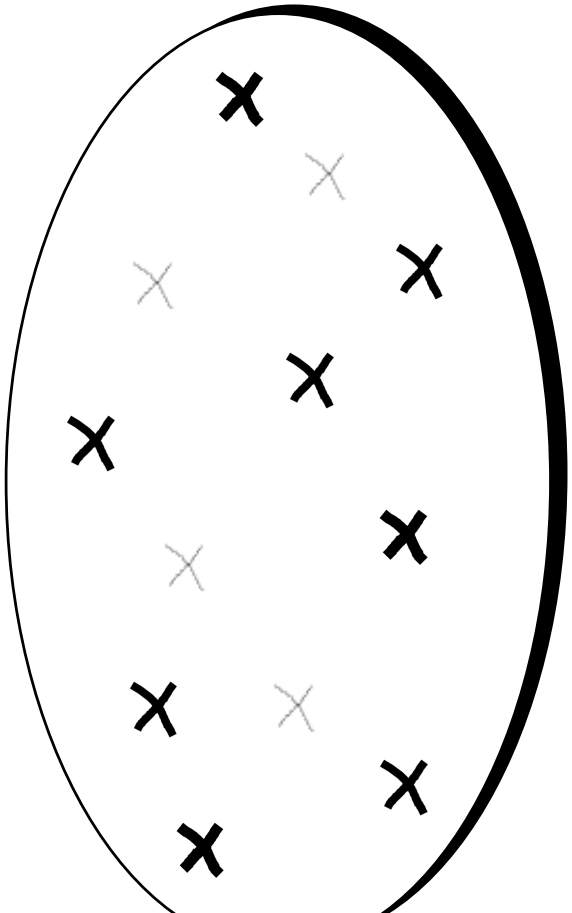
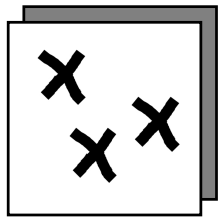
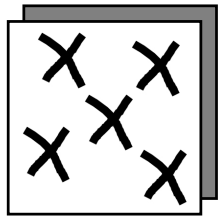




Cat. 1



Cat. 2

Cat. k

- In sorting problems there is an absolute evaluation: the assignment of an action only takes into account the intrinsic evaluation of this action on all the criteria and does not depend on nor influence the category to which another action should be assigned.
- As for the remaining problematiques the actions are compared against each other and thus there exists a relative evaluation instead of an absolute evaluation as for the previous case.

2. Main features.
2.6. Software (Figueira et al., 2005).

- Choosing: ELECTRE I, ELECTRE IV, and ELECTRE IS.
- Ranking: ELECTRE II, ELECTRE III, and ELECTRE IV.
- Ordinal classification or sorting: ELECTRE TRI.
- New software (see later on).