ELECTRE TRI-nB:

A New Multiple Criteria Ordinal Classification Method

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Contents

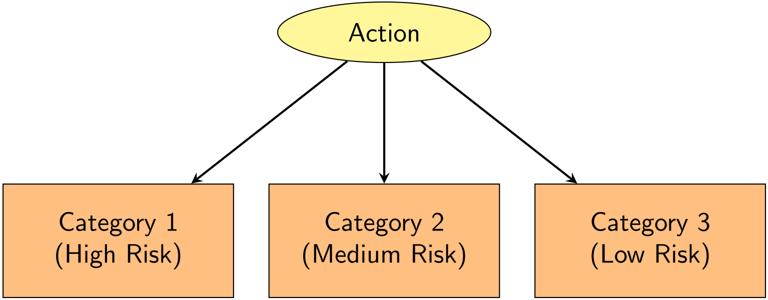
Part I

Introduction

Brief Introduction (1)

- Sorting problems are well-known in MCDA-Multiple Criteria Decision Aiding [Greco et al., 2016, Roy, 1996];
- Electre methods are well-known in the field of the outranking based approaches [Figueira et al., 2016, Figueira et al., 2013];
- Problem: Assign actions (objects), characterized according to several criteria, to pre-defined and ordered categories;
- Purpose: Present a new sorting method where boundaries are defined by sets of limiting profiles;
- It can be viewed and an extension of Electre Tri-B [Roy and Bouyssou, 1993];
- This research work is in the same line as the one done with Electre Tri-nC [Almeida-Dias et al., 2012].

Brief Introduction (2)



Part II

Some Notation and Fundamental Concepts

Notation

- x is a generic action (characterized on a set of criteria);
- A is the set of actions, not necessarily known *a priori*;
- $F = \{g_1, \ldots, g_j, \ldots, g_n\}$ is a coherent family of criteria [Roy, 1996];
- $g_j(x)$ is the performance of action x on criterion g_j;
- $C = \{C_1, \ldots, C_k, \ldots, C_q\}$ is the set of ordered categories, where C_1 is the worst;
- $B = \{B_0, \ldots, B_{k-1}, \ldots, B_{q-1}, B_q\}$ is a collection of sets of limiting profiles, where B_0 (resp. B_q) is a single lower (resp. upper) limiting profile of C_1 (resp. C_q) chosen as in Electre Tri-B;
- $B_k = \{b_{k,1}, \ldots, b_{k,j}, \ldots, b_{k,|B_k|}\}$ is the generic set of limiting profiles;
- $\sigma(x, y)$ is the outranking credibility degree (in which x outranks y);
- λ is the cutting level allowing for build a crisp relation from the fuzzy numbers $\sigma(x, y)$.

Definition 1 ($\lambda-$BINARY RELATIONS).

Given the credibility degree, $\sigma(x,y)$, and a cutting level, $\lambda \in]0.5,1]$, the following crisp binary relations are defined:

i) $xS^{\lambda}y$ iff $\sigma(x,y) \geqslant \lambda$ ($\lambda-$outranking);

ii) $x \succ^{\lambda} y$ iff $\sigma(x,y) \geqslant \lambda$ and $\sigma(y,x) < \lambda$ ($\lambda-$preference);

iii) $xI^{\lambda}y$ iff $\sigma(x,y) \geqslant \lambda$ and $\sigma(y,x) \geqslant \lambda$ ($\lambda-$indifference);

iv) $xR^{\lambda}y$ iff $\sigma(x,y) < \lambda$ and $\sigma(y,x) < \lambda$ ($\lambda-$incomparability).

Definition 2 (Basic assumptions of the set B).

The boundaries between C_k and C_{k+1} are characterized by a set of limiting profiles B_k such that, for $k = 1, \ldots, q-1$:

i) Category C_k is characterized by a set of upper limiting profiles, B_k, and by a set of lower limiting profiles, B_{k-1}; by hypothesis the elements $b_{k,j}$ of B_k belongs to C_{k+1} (this hypothesis states that categories are bounded from below);

ii) For all ordered pairs $(b_{k,j}, b_{k,i})$, such that $b_{k,j}, b_{k,i} \in B_k$, there is no strict preference between $b_{k,j}$ and $b_{k,i}$ (this implies that we have either $b_{k,j} I^{\lambda} b_{k,i}$ or $b_{k,j} R^{\lambda} b_{k,i}$);

iii) For all ordered pairs $(b_{h,j}, b_{k,i})$ such that $b_{k,j} \in B_k$ and $b_{h,i} \in B_h$ $(k > h)$ we cannot have $b_{h,i} \succ^{\lambda} b_{k,j}$.

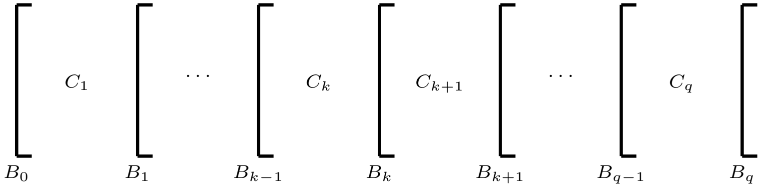


Figure: Categories bounded from below

Definition 3 (SEPARABILITY CONDITIONS ON THE SET B).

i) Dominance-based separability condition:
- Primal: For any limiting profile $z \in B_h$, with $h < k$, there is at least a limiting profile $w \in B_k$ such that wDz;
- Dual: For any limiting profile $z \in B_h$, with $h > k$, there is at least a limiting profile $w \in B_k$ such that zDw.

ii) Preference-based separability condition:
- Primal: For any limiting profile $z \in B_h$, with $h < k$, there is at least a limiting profile $w \in B_k$ such that $w \succ^{\lambda} z$;
- Dual: For any limiting profile $z \in B_h$, with $h > k$, there is at least a limiting profile $w \in B_k$ such that $z \succ^{\lambda} w$.

iii) Hyper-separability condition:
- The dominance-based separability condition holds;
- The preference-based separability condition holds.

Part III

Electre Tri-nB

Definition 4 ($\lambda-$Relations Between an Action and a Set).

i) $xS^{\lambda}B_k$ iff, for all $b_{k,j} \in B_k$, we have either $xR^{\lambda}b_{k,j}$ or $xS^{\lambda}b_{k,j}$; the latter relation being fulfilled by at least one $b_{k,j} \in B_k$ (for all $b_{k,j} \in B_k$ we cannot have $b_{k,j} \succ^{\lambda} x$);

ii) $x \succ^{\lambda} B_k$ iff, for all $b_{k,j} \in B_k$, we have either $xR^{\lambda}b_{k,j}$ or $xI^{\lambda}b_{k,j}$ or, $x \succ^{\lambda} b_{k,j}$; the latter relation being fulfilled by at least one $b_{k,j} \in B_k$ (for all $b_{k,j} \in B_k$ we cannot have $b_{k,j} \succ^{\lambda} x$);

iii) $B_kS^{\lambda}x$ iff, for all $b_{k,j} \in B_k$, we have either $b_{k,j}R^{\lambda}x$ or $b_{k,j}S^{\lambda}x$; the latter relation being fulfilled by at least one $b_{k,j} \in B_k$ (for all $b_{k,j} \in B_k$ we cannot have $x \succ^{\lambda} b_{k,j}$);

Definition 4 ($\lambda-$Relations Between an Action and a Set (cont.)).

iv) $B_k \succ^\lambda x$ iff, for all $b_{k,j} \in B_k$, we have either $b_{k,j} R^\lambda x$ or $b_{k,j} I^\lambda x$ or $b_{k,j} \succ^\lambda x$; the latter relation being fulfilled by at least one $b_{k,j} \in B_k$ (for all $b_{k,j} \in B_k$ we cannot have $x \succ^\lambda b_{k,j}$);

v) $x I^\lambda B_k$ iff, for all $b_{k,j} \in B_k$, we have either $x R^\lambda b_{k,j}$ or $x I^\lambda b_{k,j}$; the latter relation being fulfilled by at least one $b_{k,j} \in B_k$ (since it is a symmetric relation, we cannot have $b_{k,j} \succ^\lambda x$ or $x \succ^\lambda b_{k,j}$);

vi) $x R^\lambda B_k$ iff, for all $b_{k,j} \in B_k$, we have $x R^\lambda b_{k,j}$ or when $x \succ^\lambda b_{k,j}$ for some j and $b_{k,i} \succ^\lambda x$ for some i different from j (some examples showed this case can exist) (since it is a symmetric relation, we cannot have $x S^\lambda B_k$ or $B_k S^\lambda x$).

Definition 5 (Pseudo-Conjunctive (Descending) Procedure)**.**

Given the chosen $\lambda \in]0.5, 1]$:

i) Compare x with sets B_h, for $h = q-1, \ldots, 0$;

ii) Let B_{k-1} the first set such that:

   1) $xS^{\lambda}B_{k-1}$;
   2) There is no $h < k-1$ such that $B_h \succ^{\lambda} x$;

iii) Assign x to category C_k.

Definition 6 (Pseudo-Disjunctive (Ascending) Procedure).

Given the chosen $\lambda \in]0.5, 1]$:

i) Compare x with sets B_h, for $h = 1, \ldots, q$;

ii) Let B_k the first set such that:

 1) $B_k \succ^{\lambda} x$;
 2) There is no $h > k$ such that $x S^{\lambda} B_h$;

iii) Assign x to category C_k.

Proposition 1 (Relationship Between the Two Procedures)**.**

Let C_{k^*} and $C_{k^{**}}$ denote the categories $x \in A$ is assigned to, respectively, by the pseudo-conjunctive procedure and by the pseudo-disjunctive procedure. Then, we have $k^* \leqslant k^{**}$.

Proposition 2 (Electre Tri-nB becomes Electre Tri-B)**.**

If $|B_h| = 1$, for $h = 1, \ldots, q$, and the (primal or dual) dominance-based separability condition (Definition $3.i$-primal or $3.i$-dual) holds, then Electre Tri-nB corresponds to Electre Tri-B.

Part IV

Structural Properties

Definition 7 (Merging and Splitting Operations).

Consider the two operations:

a) *Merging*: Two consecutive categories, C_k and C_{k+1}, will be merged to become a new one C'_k. This is achieved by removing the limiting profile B_k. The category C'_k is bounded by the sets B_{k-1} and B_{k+1};

b) *Splitting*: The category C_k will be split into two new consecutive categories C'_k and C'_{k+1}. This is achieved by inserting a new limiting profile B'_k, such that the elements of B'_k fulfill the properties stated in Definition 2.

Definition 8 (Structural Requirements).

The following are natural requirements an assignment procedure must fulfill:

i) *Uniqueness*: Each action is assigned to a unique category;

ii) *Independence*: The assignment of an action does not depend on the assignment of the other actions;

iii) *Conformity*:

  a) If $xS^{\lambda}B_k$ and $B_{k'} \succ^{\lambda} x$ $(k' > k)$, then action x is assigned to C_f with $k+1 \leqslant f \leqslant k'$;

  b) Each limiting profile $b_{k,j} \in B_k$ is assigned to C_{k+1};

iv) *Monotonicity*: If an action x dominates an action y, xDy, and if y is assigned to C_k, then x is assigned to $C_{k'}$ with $k' \geqslant k$;

v) *Homogeneity*: If two actions compare the same way with respect to the limiting profiles, they must be assigned to the same category;

vi) *Stability*: Categories are stable w.r.t. merging and splitting operations.

Theorem 1 (Structural Properties).

- Under the basic assumptions (Definition 2) the assignment procedures fulfill the requirements of uniqueness, independence, homogeneity, and monotonicity.
- Adding only the primal dominance-based separability condition (Definition $3.i-$primal) Part $a)$ of conformity holds.
- With the dominance-based separability condition (Definition $3.i$) the requirement of stability is verified.
- Under the basic assumptions (Definition 2) and the preference-based separability condition (Definition $3.ii$) Part $b)$ of conformity holds.
- Under the hyper-separability condition (Definition $3.iii$) all the structural requirements are fulfilled. (hyper-separability is less restrictive as the all-to-all complete dominance.)

For the proof see [Fernández et al., 2017].

Part V

A Numerical Example

Numerical Example (1)

- Action x is characterized on four criteria $g(x) = [6, 1, 2, 1]$;
- The ordered set of three categories $C = \{C_1, C_2, C_3\}$ (C_3 is the most preferred);
- Weights: $w_j = 0.25$, for $j = 1, \ldots, 4$;
- Indifference thresholds: $q_j = 0.5$, for $j = 1, \ldots, 4$;
- Preference thresholds: $p_j = 1$, for $j = 1, \ldots, 4$;
- Veto thresholds: $v_j = 2.5$, for $j = 1, \ldots, 4$;
- Cutting level: $\lambda = 0.75$.

Numerical Example (2)

- Start with Electre Tri-B;
- Performances of limiting profiles b_k:

$b_k/g_j(\centerdot)$	$g_1(\centerdot)$	$g_2(\centerdot)$	$g_3(\centerdot)$	$g_4(\centerdot)$
b_0	1	1	1	1
b_1	2	2	2	2
b_2	3	3	3	3
b_3	6	6	6	6

- $\lambda-$binary relations: $\succ^{\lambda}\ R^{\lambda}\ R^{\lambda}\ \succ^{-1\lambda}$;
- Assignments of x: Pseudo-conjunctive (C_1) and Pseudo-disjunctive (C_3).

- Add a new profile between C_1 and C_2 w.r.t. the initial situation: $g(b_{1,2}) = [3.2, 2.8, 1, 1]$;
- $\lambda-$binary relations: $\succ^{\lambda} \succ^{\lambda}\ R^{\lambda}\ \succ^{-1\lambda}$;
- Assignments of x: Pseudo-conjunctive (C_2) and Pseudo-disjunctive (C_3);

- Add a new profile between C_2 and C_3 w.r.t. the initial situation: $g(b_{2,2}) = [4, 3, 3, 2]$;
- $\lambda-$binary relations: $\succ^{\lambda}\ R^{\lambda}\ \succ^{-1\lambda} \succ^{-1\lambda}$;
- Assignments of x: Pseudo-conjunctive (C_1) and Pseudo-disjunctive (C_2);

- All together now;
- $\lambda-$binary relations: $\succ^{\lambda} \succ^{\lambda} \succ^{-1\lambda} \succ^{-1\lambda}$;
- Assignments of x: Pseudo-conjunctive (C_2) and Pseudo-disjunctive (C_2).

Part VI

Conclusions

Conclusions

- Inspired on the success of Electre Tri-B, we presented in this talk a new multiple criteria sorting method, called Electre Tri-nB;
- It gives new possibilities to the DM for characterizing the limiting boundaries between adjacent categories;
- When each boundary is characterized by a single limiting profile, Electre Tri-nB does not differ from Electre Tri-B;
- The new method is especially recommended when a single limiting profile is not sufficient for a good characterization of its associated boundary;
- In Electre Tri-nB, each limiting profile added to the description of a boundary is a new piece of information that helps to an improved characterization of such boundary;
- The example shows how Electre Tri-nB works, and why it can suggest more appropriate assignments when the boundaries are enhanced with additional limiting profiles.

Part VII

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